# Looking at the small things: applying rigor to idle thoughts

Denali Relles, Zachary Feng

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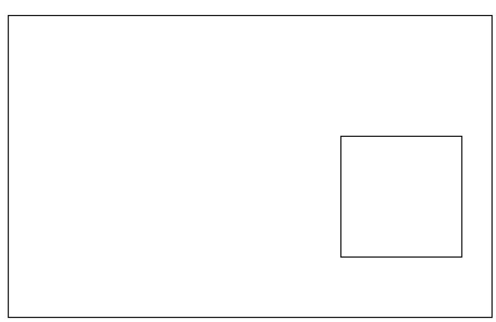
I'm a math major, so it might not surprise you that I enjoy thinking about math all the time. I look around at the things in my vicinity, and I try to reason about their geometric properties. Everyone does this to some extent, perhaps explicitly when trying to move furniture into a new house or trying to wash a pan that doesn't quite fit in the sink, or perhaps more subtly, which I don't have an example of. This is an example of a small thought that turned into a problem that was not entirely trivial.

## The setup

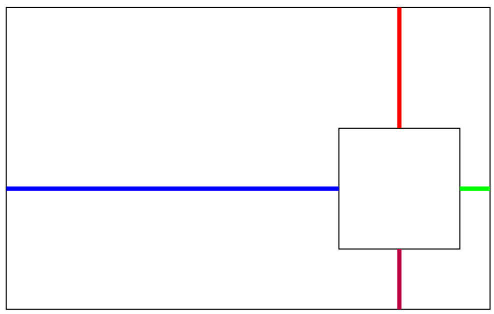
What's the most common geometric shape you can see from where you are? There's not a bad chance that it's the rectangle. They're everywhere. Doors, windows, apartment buildings, books, floorboards, tiles, and those are just the ones I can see from where I'm sitting.

So picture your favorite rectangle. Now imagine I put a square in it, like this:



This picture is already annoying me. I don't know about you, but the asymmetry is just a little unsettling. It also breaks the rule I'm going to explain next. We could just put the square in the middle. Nice and symmetrical, right? But I have a sneakier plan.

There's four sides to a rectangle, and four sides to a square. That means there's four distances between them. Here they are colored in:

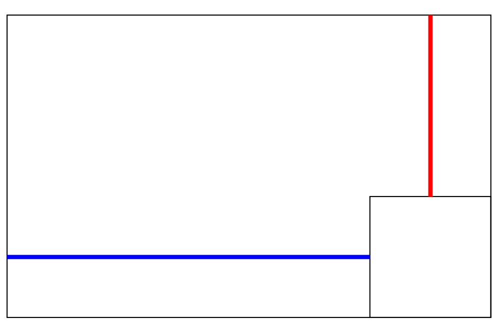


And when I have some things, I want to put them in order. So: is there a place to put the square where the lengths increase as you go around? That is, purple less than green less than green less than red less than blue? Note that the diagram above does not work, this is on purpose.

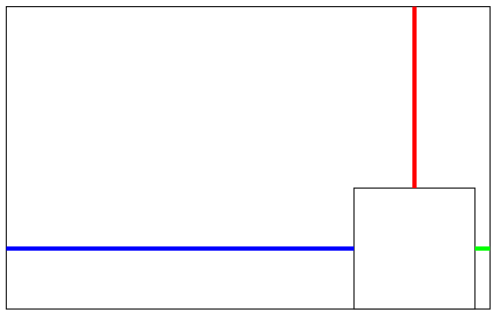
I've thought about this in the past, but I usually dismiss any further thinking with the answer "probably." But since my friend Zachary Feng enjoys solving puzzles with me, big or small, I decided to tell him about it and figure out a really good answer.

More formally, I will clarify: let $a, b, c$ be real numbers with $a < b < c$. The square has all sides length $a$ and the rectangle has sides length $b$ and $c$.

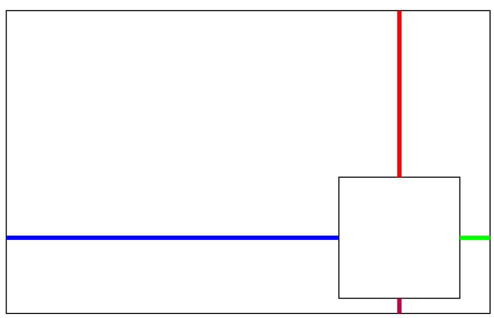
I gave zach this problem, and he had a quick answer. I mean, this isn't a hard problem but here's what he said. Start with the square in the corner like this:



Blue is already bigger than red! we're doing good. Since we are given $a < c$, there must be a "tiny" amount we can move the square to the left. "Tiny" as in, say, one five hundreth the space we have. It's exaggerated in the picture below:



Now we could say we're done, but I'd like the square to not be touching any of the sides. So we pick another "tiny" distance, again like one five hundreth of the space we have, and move the square diagonally up and to the left by the same tiny amount. Observe:



So the blue side is still bigger than the red side, because it started out that way and we only moved the square a "tiny" bit. The red side is bigger than the green one because the red one has some length while the green side is "tiny". And the green side is longer than the purple because the green one is two tinys while the purple is only one "tiny".

I swear this is rigorous. For example, I can define "tiny" to be exactly the following: one fourth of whichever is smaller, $b-a$ or $c-b$. This is math: $\varepsilon = \frac{1}{4}\min\{b-a, c-b\}$. Then in the end the distance we have are:

$$c - a - 2\varepsilon \qquad b - a - \varepsilon \qquad 2\varepsilon \qquad \varepsilon$$

. Here's how it works.

Since $\varepsilon \leq \frac{1}{4}(c-b)$:

$$\begin{aligned} c-a-2\varepsilon &\geq c-a-\frac{1}{2}(c-b) \\ &> c-a-(c-b) \\ &= b-a \\ &> b-a-\varepsilon \end{aligned}$$

Since $4\varepsilon \leq (b-a)$:

$$\begin{aligned} b-a-\varepsilon &\geq 4\varepsilon-\varepsilon \\ &= 3\varepsilon \\ &> 2\varepsilon \end{aligned}$$

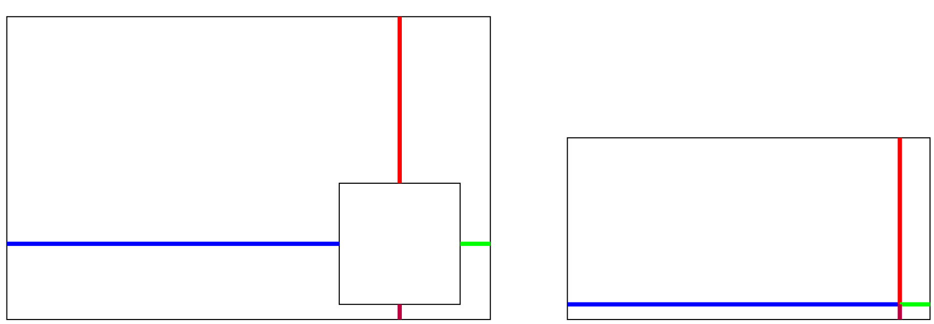
As for $2\varepsilon > \varepsilon$, I think that's obvious.

So we're done! I've taken this simple musing about rectangles and squares, and applied to math to it, and the answer is yes!

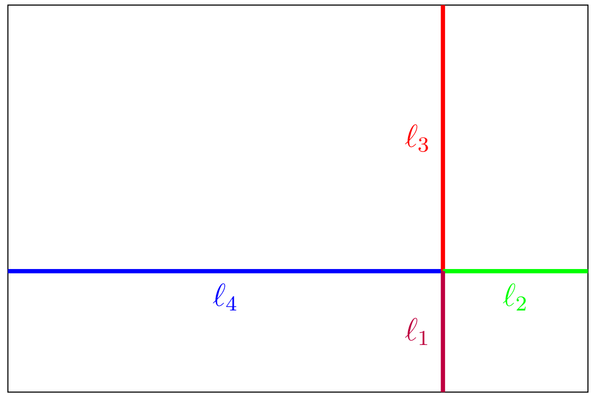
## We need to go deeper

Or we could keep going. Sure, increasing is nice, but what if there was a little more structure? Can we make... a geometric series?

First we realized that this problem is equivalent to a rectangle with sides $b-a$ and $c-a$ with a point in it. This just makes everything a little simpler. Observe:



And I will label the lines, $\ell_1 \ldots \ell_4$, in increasing order:

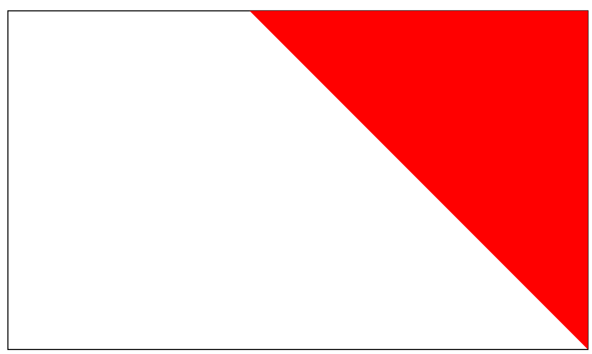



What I mean by geometric series is:

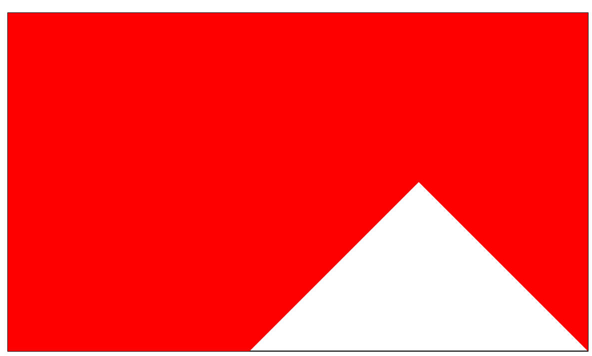
$$\frac{\ell_1}{\ell_2} = \frac{\ell_2}{\ell_3} = \frac{\ell_3}{\ell_4}$$

A common ratio.

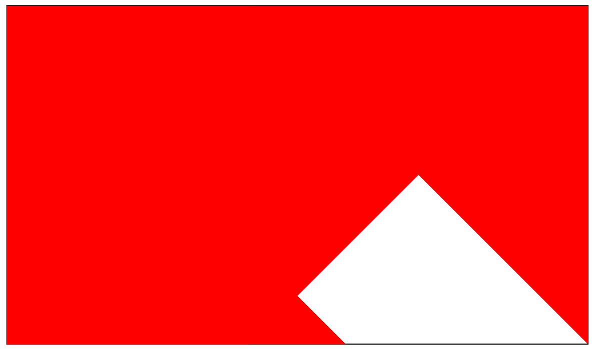
But first, what are all possible points that satisfy the increasing-order thing? Well, first we need $\ell_1 < \ell_2$, so that rules out some places:



Then we need $\ell_2 < \ell_3$:



Finally, $\ell_3 < \ell_4$:



A couple things. This restricts the entire left half and the entire top half, which makes sense because $\ell_1 < \ell_3$ and $\ell_2 < \ell_4$. That's good. It's good when things make sense. Another note: if the rectangle is much longer than it is wide (at least twice as wide as it is tall) the last restriction won't actually do anything.

Now for what we want. We want the ratios to be equal. We had a couple approaches. We thought about maps. Like, the function that takes the point in the rectangle, which can be thought of as a map $f : \mathbb{R}^2 \to \mathbb{R}^4$:

$$f(x, y) = (x, y, a - x, b - y)$$

Or the map that represents taking ratios, $g : \mathbb{R}^4 \to \mathbb{R}^3$

$$g(\ell_1, \ell_2, \ell_3, \ell_4) = \left( \frac{\ell_1}{\ell_2}, \frac{\ell_2}{\ell_3}, \frac{\ell_3}{\ell_4} \right)$$

Maybe their composition:

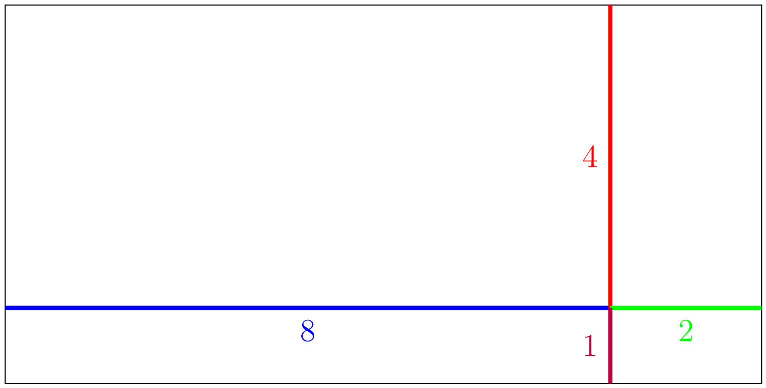
$$g \circ f(x, y) = \left( \frac{x}{y}, \frac{y}{a - x}, \frac{a - x}{b - y} \right)$$

We could look for this funciton intersection the line $x = y = z$ in $\mathbb{R}^3$. We tried that. We looked at another map, $h : \mathbb{R}^4 \to \mathbb{R}^2$

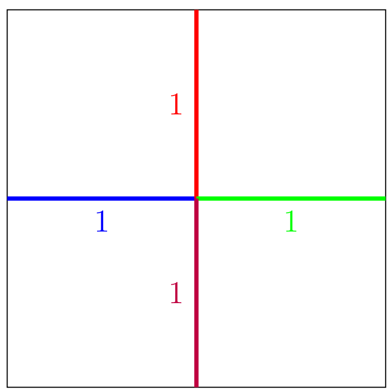
$$g(\ell_1, \ell_2, \ell_3, \ell_4) = \left( \frac{\ell_1}{\ell_2} - \frac{\ell_2}{\ell_3}, \frac{\ell_2}{\ell_3} - \frac{\ell_3}{\ell_4} \right)$$

Like, the differences between the ratios. We want this to evaluate to $(0, 0)$. Is there some way this could use the mean value theorem? Could I use the technique from that one 3blue1brown video where you find a loop that maps to a loop around the origin?

Zach came up with this example:




Which clearly works. It's also worth noting this example *technically* works, in that it makes a geometric progression:




Zach came up with the final answer. He saw that $\ell_1$ as well be 1. Then, everything depends of your choice for $\ell_2$. If $\ell_1 = 1$ and $\ell_2 = y$, then the sequence must go $1, y, y^2, y^3$ to be geometric. The rectanlge these four lengths fit in is the one with side lengths $1 + y^2$ and $y + y^3$. And the ratio between these sides is $(y + y^3)/(1 + y^2)$ which happens to be exactly $y$. What this tells us is that for any given side ratio, we can make a rectangle with this side ratio and put a point in it that does this geometric sequence thing. In fact, it tells us that there is a unique point for every rectangle, because the aspect ratio is uniquely determined by the choice of $y$, the second length.

Maybe this was all obvious. Maybe I explained it terribly. But at least now I know the answer.